

SPECIAL AVERAGING METHOD FOR SOLVING SOME 3-D HEAT TRANSFER PROBLEMS

Aivars Aboltins¹, Ilmars Kangro², Harijs Kalis³

¹Latvia University of Life Sciences and Technologies, Latvia;

²Rezekne Academy of Technologies, Latvia;

³Institute of Mathematics and Computer Science, University of Latvia, Latvia

aivars.aboltins@llu.lv, ilmars.kangro@rta.lv, harijs.kalis@lu.lv

Abstract. We consider the special conservative averaging method for solving the heat transfer boundary-value problem in the 3-D domain. Looking back to the history of mathematics, integral parabolic splines related to the conservative averaging method (CAM), in 1980s A. Buikis developed CAM for partial differential equations with discontinuous coefficients in environments with a layered structure. We consider newly designed for CAM the special integral hyperbolic type splines for solving some 3-D heat transfer initial-boundary value problems. Using these types of splines, the present problem is reduced to the 2-D problem with respect to one coordinate. This procedure also allows reducing the 2-D problem to 1-D initial-boundary value problem and to initial value problem for ordinary differential equations. Thus, the solution to the problem can be obtained analytically. The solution of corresponding 3-D initial-boundary value problem is obtained also numerically, using for approach differential equations of discretization in space applying the differences. This method can be considered as a generalization of the method of finite volumes, because it can solve BVP with convection, not only in z-direction, but also in y-direction and x-direction, based on the CAM, and, at the same time, through transformation, moving to the obtained BVP without convective terms. The approximation of the 3-D nonstationary problem is based on the implicit finite-difference and alternating direction (ADI) methods. The numerical solution is compared with the spline solution. The publication provides a wide-ranging insight into the practical uses of 3-D heat transfer boundary-value problems in general, as well as the effectiveness of the CAM application, in conjunction with the ADI method, is analyzed.

Keywords: 3-D problem, PDE, heat transfer, averaging method, splines, analytical solution.

Introduction

During the modelling of thermal systems we have to, due to their complexity, work with a number of simplifications. In practice, however, these simplifications have no profound effect on the model's accuracy. During the creation of regulation we have to account for the slow dynamic of the system, determine which parameters affect individual heat flows and adjust accordingly.

Heat transfer problems and their mathematical modelling are well known. As the need for a more precise description of the process, a more accurate mathematical model increases the difficulty of solving this mathematical problem. Many real convective heat transfer problems have been described mathematically and an analysis of their numerical results has been given [1]. Assumptions are often used to solve these problems, which simplify the problem itself and thus the solution.

On the other hand, using a mathematical apparatus, it is possible to simplify the mathematical description of the problems under consideration and solve them with simplified, well-known methods, and to evaluate the accuracy of these solutions. This is especially true for solving heat transfer problems in multilayer environments under changing process conditions [2; 3]. Another area of widespread concern is the 3-D environment. For the most part, problems are considered in 1-D, 2-D environments or symmetric environments, where it is easier to find a solution. Boiling heat transfer during fluid flow is analyzed at [4]. Two dimensional temperature fields are determined at this case. 3-D heat transfer through cylindrical structures (filled with oil) with irregular cross-sections, which are bounded by a homogeneous elastic medium and heated, is described in [5]. Here 3-D problem can be computed as a summation of 2-D solutions. The natural-convection thermal performance of a dielectric liquid in a cubical module triggered by a varying rectangular heat generation source is numerically examined at [6]. Researchers use programming packages very widely. 3-D case of natural convection inside a rectangular enclosure with a discretely heated sidewall has been investigated numerically and experimentally with ANSYS in [7]. The mathematical model for liquid film flow in 3-D formulation has been developed with consideration of thickness changes in vertical and horizontal directions [8].

3-D convection-diffusion problem is considered in [9]. It appears that the solution to the problem requires substantial mathematical collateral resources, both in theoretical terms (fourth-order alternating

direction implicit scheme) and in terms of the use of numerical calculation algorithms (three-dimensional convection-diffusion equation). Metal distribution in peat layer blocks is modelled in [10].

The task of sufficient accuracy numerical simulation of quick solution of 3-D problems for mathematical physics is important in known areas of the applied sciences. To achieve this goal for engineering calculation the conservative averaging method (CAM) is chosen by using special integral type splines. The application of this method in modelling the combustion process is shown [11].

It is a necessity to solve the 3-D initial-boundary-value problems for parabolic type partial differential equations of second order with constant parameters. The special spline which interpolates middle integral values of a piece-wise smooth function is defined. A. Buikis considered CAM with parabolic type splines for mathematical simulation of the mass transfer process in multilayered underground systems [12]. The theoretical basis of the special parabolic, hyperbolic and exponential type splines with their practical applications has been developed in [13-15].

The conservative averaging method has been applied in the technical sphere, modelling the heat distribution in the 3-D area of automotive fuse [16]. A cylindrical mathematical model of automotive fuse to characterize the heat-up process in the fuse is described by partial differential equations of transient heat conduction. CAM with integral parabolic type splines has been used to get the approximated solution of studied problem with analytical formulas [17]. The conservative averaging-reconstruction method (Ring Average) is used for MHD solvers applying explicit finite-volume methods for solving MHD flow problems in cylindrical/spherical geometries. Here, the volume-averaged conservative fluid variables (mass, momentum, energy) are usually defined at cell centres. Thus, the averaging-reconstruction of the fluid variables is essentially a one-dimensional process along the azimuthal direction, and therefore the conservative fluid variables in each ring have been reconstructed by the piecewise parabolic method [18]. Here CAM for special hyperbolic type splines is developed. With the help of these splines the 3-D initial-boundary value (IBV) of the heat transfer problem with respect to one coordinate is reduced to 2-D and 1-D IBV problems. Unlike previous investigations, here we use transformation in PDEs for reducing the convection-diffusion heat transfer problem to the problem without convection terms. The solution of the corresponding 3-D initial-boundary value problem is obtained also numerically, using the implicit finite-difference and alternating direction (ADI) methods of J. Douglas and H. H. Rachford [19].

These splines in every direction of averaging contain parameters, where it can be chosen so that the error of the solution is decreasing. In the limit case, when the spline parameters tend to zero, we get the integral parabolic spline, obtained from A. Buikis [12].

1. Mathematical model

The nonstationary convection-diffusion process of heat transfer is considered in 3-D parallelepiped

$$\Omega = \{(x, y, z): 0 \leq x \leq L_x, 0 \leq y \leq L_y, 0 \leq z \leq L_z\}.$$

We will find the distribution of the temperature $u = u(x, y, z, t)$, °C at the point $(x, y, z, t) \in \Omega$ and at the time t from the following 3-D initial-boundary value problem (IBVP) for partial differential equation (PDE) with convection in z direction:

$$\left\{ \begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} \left(D_x \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(D_y \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(D_z \frac{\partial u}{\partial z} \right) + r_z \frac{\partial u}{\partial z}, \\ x &\in (0, L_x), y \in (0, L_y), z \in (0, L_z), t \in (0, t_f), \\ \frac{\partial u(0, y, z, t)}{\partial x} &= \frac{\partial u(x, 0, z, t)}{\partial y} = \frac{\partial u(x, y, 0, t)}{\partial z} = 0, \\ D_x \frac{\partial u(L_x, y, z, t)}{\partial x} &+ \alpha_x (u(L_x, y, z, t) - u_{ax}) = 0, \\ D_y \frac{\partial u(x, L_y, z, t)}{\partial y} &+ \alpha_y (u(x, L_y, z, t) - u_{ay}) = 0, \\ D_z \frac{\partial u(x, y, L_z, t)}{\partial z} &+ \alpha_z (u(x, y, L_z, t) - u_{az}) = 0, u(x, y, z, 0) = u_0(x, y, z), \end{aligned} \right. \quad (1)$$

where D_x, D_y, D_z – constant heat diffusion coefficients, $m^2 \cdot s^{-1}$;
 $\alpha_x, \alpha_y, \alpha_z$ – constant heat transfer coefficients in the 3 kind boundary conditions, $m \cdot s^{-1}$;
 u_{az}, u_{ay}, u_{ax} – are the given temperatures on the boundaries, $^{\circ}C$;
 r_z – constant convective velocity in the z-direction, $m \cdot s^{-1}$;
 t_f – final time, s;
 $u_0(x, y, z)$ – given initial temperature, $^{\circ}C$.

It should be said that themes of tasks/problems solved by PDE (also the above mentioned convection-diffusion IBVP (1)) include different, often very distinguished areas of science and technique, such as economics, fluid dynamics, forecasting, astrophysics, oceanography, meteorology, etc., [20; 21].

Using the transformation $u(x, y, z, t) = \exp(-r_v z)v(x, y, z, t)$, $r_v = r_z/2D_z$ we can reduce the problem (1) to the problem without a convective term:

$$\left\{ \begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial}{\partial x} \left(D_x \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(D_y \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(D_z \frac{\partial v}{\partial z} \right) - D_z r_v^2 v, \\ x &\in (0, L_x), y \in (0, L_y), z \in (0, L_z), t \in (0, t_f), \\ \frac{\partial v(0, y, z, t)}{\partial x} &= \frac{\partial v(x, 0, z, t)}{\partial y} = 0, \frac{\partial v(x, y, 0, t)}{\partial z} - r_v v(x, y, 0, t) = 0, \\ D_x \frac{\partial v(L_x, y, z, t)}{\partial x} &+ \alpha_x (v(L_x, y, z, t) - \exp(r_v z)u_{ax}) = 0, \\ D_y \frac{\partial v(x, L_y, z, t)}{\partial y} &+ \alpha_y (v(x, L_y, z, t) - \exp(r_v z)u_{ay}) = 0, \\ D_z \frac{\partial v(x, y, L_z, t)}{\partial z} &- 0.5r_z v(x, y, L_z, t) + \alpha_z (v(x, y, L_z, t) - u_{az} \exp(r_v L_z)) = 0, \\ v(x, y, z, 0) &= u_0(x, y, z) \exp(r_v z). \end{aligned} \right. \quad (2)$$

2. The CAM with the exponential and hyperbolic type integral spline for 1-D initial boundary-value problem

For motivation of the transformation method we especially consider the 1-D IBV problem with convection:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial z} \left(D_z \frac{\partial u}{\partial z} \right) + r \frac{\partial u}{\partial z} + F, \quad z \in (0, L_z), t \in (0, t_f), \\ D_z \frac{\partial u(0, t)}{\partial z} - \beta(u(0, t) - u_0) = 0, \\ D_z \frac{\partial u(L_z, t)}{\partial z} + \alpha(u(L_z, t) - uL) = 0, \quad u(z, 0) = 0, \end{cases} \quad (3)$$

where F – fixed constant, °C·s⁻¹;
 r – fixed constant, m·s⁻¹.

We are looking at the convection term of the PDV in only one z -direction to more conveniently show how after transformation a new PDV that does not contain a convective term can be transitioned and the dimension can be further reduced by using a hyperbolic spline with a known parameter. If initially the PDV contains also convective terms in other directions (for example, x, y), then the process of dimensional reduction can be analogously continued by obtaining ordinary differential equations.

It should be noted that there are enough processes in nature, where the dominant mass or temperature transfer takes place only in the direction of one coordinate, such as z . Therefore, physical processes such as filtration, calculation of contamination in underground waters, as well as certain types of combustion processes and other processes that are subject to the given mathematical model may also be modelled through the problem under consideration.

We use without transformation the following exponential type spline approximation:

$$u(z, t) = u_z(t) + m_z(t)(z - L_z/2) + e_z(t)(\exp(a_z(z - L_z/2)) - q),$$

where $q = (2/a_z L_z) \sin(a_z L_z/2)$, $u_z(t) = (L_z)^{-1} \int_0^{L_z} u(z, t) dz$ – averaged value.

The unknown functions m_z, e_z are determined from boundary conditions by $z = 0, z = L_z$:

$$m_z + e_z b_m - \beta_1(u_z - 0.5m_z L_z + e_z(\exp(-a_z L_z/2) - q) - u_0) = 0,$$

$$m_z + e_z b_{mp} + \alpha_1(u_z + 0.5m_z L_z + e_z(\exp(a_z L_z/2) - q) - uL) = 0,$$

where $\beta_1 = \beta/D_z, \alpha_1 = \alpha/D_z, b_m = a_z \exp(-a_z L_z/2), b_p = a_z \exp(a_z L_z/2), a_z = -r/D_z$.

Therefore

$$e_z = a_2 u_z + b_2; \quad m_z = a_1 u_z + b_1, \quad a_1 = (\beta_1 a_{22} + \alpha_1 a_{12}) / \det, \quad a_2 = -(\beta_1 a_{21} + \alpha_1 a_{11}) / \det,$$

$$b_1 = -(\beta_1 a_{22} u_0 + \alpha_1 a_{12} uL) / \det, \quad b_2 = (\beta_1 a_{21} u_0 + \alpha_1 a_{11} uL) / \det; \quad \det = a_{11} a_{22} - a_{12} a_{21},$$

$$a_{11} = 1 + 0.5\beta_1 L_z, \quad a_{21} = 1 + 0.5\alpha_1 L_z, \quad a_{12} = b_m - \beta_1(\exp(-a_z L_z/2) - q);$$

$$a_{22} = b_p + \alpha_1(\exp(a_z L_z/2) - q).$$

Using the integral averaging from equation (3), we have the initial value problem in the following form:

$$\begin{cases} \frac{\partial u_z(t)}{\partial t} = a_z e_z(t) q (D_z a_z + r) + m_z(t) r + F, \quad t \in (0, t_f), \\ e_z(t) = a_2 u_z(t) + b_1, m_z(t) = a_1 u_z(t) + b_1, u_z(0) = 0, t \in (0, t_f). \end{cases} \quad (4)$$

We have the following solution of these ODEs

$$u_z(t) = \frac{B_0}{A_0} (\exp(A_0 t) - 1),$$

where $A_0 = a_z a_2 q (D_z a_z + r) + r a_1, B_0 = a_z b_2 q (D_z a_z + r) + r b_1$.

The 1-D stationary convection-diffusion boundary-value problem (BVP) in the multi-layered plane domain in z -direction with different convection velocity in every layer is considered in [11]. In this

case, the solution to the BVP is discontinuous and can easily be obtained with the conservative averaging method, using the exponential type spline function.

Using the transformation

$$u(z, t) = \exp(-r_v z)v(z, t), \quad r_v = r / 2D_z,$$

we can reduce the problem (3) to the problem without the convective term:

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial}{\partial z} \left(D_z \frac{\partial v}{\partial z} \right) - D_z r_v^2 v + F \exp(r_v z), \quad z \in (0, L_z), t \in (0, t_f), \\ \frac{\partial v(0, t)}{\partial z} - r_v v(0, t) - \beta_1 (v(0, t) - u_0) = 0, \\ \frac{\partial v(L_z, t)}{\partial z} - r_v v(L_z, t) + \alpha_1 (v(L_z, t) - uL \exp(r_v L_z)) = 0, \quad v(z, 0) = 0. \end{cases} \quad (5)$$

We use the following hyperbolic type spline approximation:

$$u(z, t) = v_z(t) + m_z(t)f_{z1} + e_z(t)f_{z2},$$

with the following two fixed hyperbolic functions and the parameter $a_z = |r_v|$:

$$f_{z1} = \frac{0.5L_z \sinh(a_z(z - 0.5L_z))}{\sinh(0.5a_z L_z)}, \quad f_{z2} = \frac{\cosh(a_z(z - 0.5L_z)) - A_{0z}}{8 \sinh^2(0.25a_z L_z)},$$

where $A_{0z} = \frac{\sinh(0.5a_z L_z)}{0.5a_z L_z}$, $v_z(t) = \frac{1}{L_z} \int_0^{L_z} v(z, t) dz$ is the averaged value.

It can be seen, if parameters tend to zero, then in the limit case we get the integral parabolic spline from A. Buikis [12].

The unknown functions m_z, e_z are determined from the boundary conditions of (5) by $z = 0, z = L_z$:

$$d_z m_z - k_z e_z - (\beta_1 + r_v)(v_z - 0.5m_z L_z + e_z b_z) + \beta_1 u_0 = 0,$$

$$d_z m_z + k_z e_z - (\alpha_1 - r_v)(v_z + 0.5m_z L_z + e_z b_z) - \alpha_1 uL \exp(r_v L_z) = 0,$$

$$d_z = 0.5a_z L_z \coth(0.5a_z L_z), \quad k_z = 0.25a_z \coth(0.25a_z L_z), \quad b_z = \frac{\cosh(0.5a_z L_z) - A_{0z}}{8 \sinh^2(0.25a_z L_z)}.$$

Therefore,

$$e_z = a_2 u_z + b_2, \quad m_z = a_1 u_z + b_1, \quad a_1 = ((\beta_1 + r_v)a_{22} + (\alpha_1 - r_v)a_{12}) / \det,$$

$$a_2 = -((\beta_1 + r_v)a_{21} + (\alpha_1 - r_v)a_{11}) / \det, \quad b_1 = -(\beta_1 a_{22} u_0 + \alpha_1 a_{12} uL \exp(r_v L_z)) / \det,$$

$$b_2 = (\beta_1 a_{21} u_0 + \alpha_1 a_{11} uL \exp(r_v L_z)) / \det, \quad \det = a_{11} a_{22} - a_{12} a_{21}, \quad a_{11} = d_z + 0.5(\beta_1 + r_v)L_z,$$

$$a_{21} = d_z + 0.5(\alpha_1 - r_v)L_z, \quad a_{12} = -k_z - (\beta_1 + r_v)b_z, \quad a_{22} = k_z + (\alpha_1 - r_v)b_z.$$

Using the integral averaging from equation (5), we have the initial value problem in the following form

$$\begin{cases} \frac{\partial v_z(t)}{\partial t} = D_z (2k_z e_z(t) / L_z - r_v^2 v_z(t)) + F(\exp(r_v L_z) - 1) / (L_z r_v), \\ e_z(t) = a_2 v_z(t) + b_2, m_z(t) = a_1 v_z(t) + b_1, v_z(0) = 0, t \in (0, t_f). \end{cases} \quad (6)$$

We have the following solution of these ODEs

$$v_z(t) = \frac{B_0}{A_0} (\exp(A_0 t) - 1),$$

where $A_0 = D_z(2k_z a_2 / L_z - r_v^2)$, $B_0 = D_z 2k_z b_2 / L_z + F(\exp(r_v L_z) - 1) / (L_z r_v)$.

2.1. Some numerical results

The BVP (3) was solved by two methods. The first method resulted in obtaining a solution $u(z, t) = U$ using exponential type splines. When using transformation $u(z, t) = \exp(-r_v z)v(z, t)$, the BVP (3) was transformed into a BVP (5), its solution $v(z, t) = V_s$ was obtained by hyperbolic type splines. The solution of the BVP (3) by the second method was obtained by multiplying $v(z, t) = V_s$ by $\exp(-r_v z)$, i.e., $U_s = \exp(-r_v z)v(z, t)$.

Problems (3) and (5) were solved with such dimensionless parameters (their numerical values are chosen to solve practical problems), with the aim of clarifying a numerical experiment whether the solution (3) obtained by the first method and the second method does not differ significantly, that is, or the application of the transformation does not reduce the accuracy of the calculations to be carried out:

$$u_0 = 1, uL = 10, L_z = 1, \alpha = 100, \beta = 200, D_z = 10^{-2},$$

$$r = -0.01, F = 2, t_f = 4, t_j = 0.4j, j = \overline{0, 10},$$

$$z_i = \frac{1}{40}i, i = \overline{0, 40}, a_z = 0.1; 0.05.$$

The maximal values of the obtained solutions are given for comparison: $u(z, t) = U = 10.0032$ (Fig. 1), $v(z, t) = V_s = 9.534$ (Fig. 2.), $U_s = \exp(-r_v z)v(z, t) = 10.0053$.

When we solve the BVP (3) through transformation, the accuracy of the calculations is virtually not lost because the maximal error between U and U_s is 0.0403.

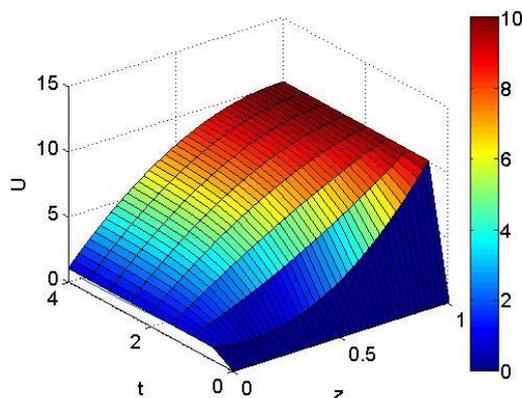


Fig. 1. Exponential type spline solution – the surface $u(z, t) = U$

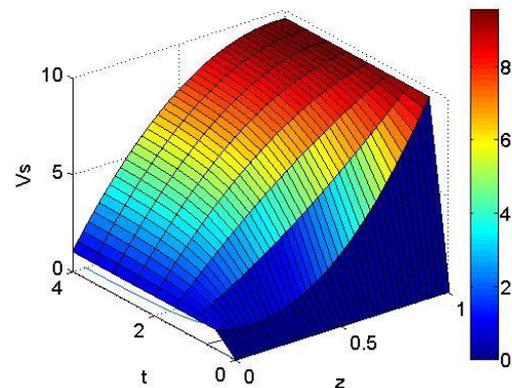


Fig. 2. Hyperbolic type spline solution – the surface $v(z, t) = V_s$

These numerical calculations in the present paper were performed to demonstrate the usefulness of the transformation by determining the parameter a_z of the hyperbolic type spline function. For practical calculations, this method could be used in the work [22], because a similar problem is considered there – the diffusion and convection filtration problem of one substance through the pores of a porous material, which may absorb and immobilize some of the diffusing substance.

The domain consists of a porous material, where incompressible liquid pollutants move in z -direction through the pores of a filter. The optimal parameters for the hyperbolic type spline function in z -direction without transformation can be obtained only with the method of iterations.

3. The CAM with the hyperbolic type integral spline approximation for solving the initial boundary-value 3-D problem

3.1. The CAM in z -direction

For solving the boundary value problem (2) for every $t > 0$ we consider the following hyperbolic type spline approximation with respect to z-direction:

$$u(x, y, z, t) = v_z(x, y, t) + m_z(x, y, t)f_{z1} + e_z(x, y, t)f_{z2},$$

with the following two fixed hyperbolic type functions

$$f_{z1}, f_{z2}, v_z(x, y, t) = \frac{1}{L_z} \int_0^{L_z} v(x, y, z, t) dz$$

is the averaged value and $a_z = r_v, a_z = r_v$ is the fixed parameter.

The unknown functions m_z, e_z are determined from the boundary conditions of (2) by $z = 0, z = L_z$:

$$d_z m_z - k_z e_z - r_v (v_z - 0.5 m_z L_z + e_z b_z) + \beta_1 u 0 = 0, \quad m_z = p_z e_z + a_1 v_z,$$

$$p_z = (k_z + r_v b_z) / (d_z + 0.5 L_z r_v),$$

$$d_z = 0.5 a_z L_z \coth(0.5 a_z L_z), \quad k_z = 0.25 a_z \coth(0.25 a_z L_z), \quad a_1 = r_v / (d_z + 0.5 r_v L_z)$$

$$D_z (d_z m_z + k_z e_z) + a_2 (c_z + 0.5 m_z L_z + e_z b_z) - \alpha_z u_{az} \exp(r_v L_z) = 0,$$

where $b_z = \frac{\cosh(0.5 a_z L_z) - A_{0z}}{8 \sinh^2(0.25 a_z L_z)}, \quad a_2 = \alpha_z - 0.5 r_z.$

Therefore,

$$e_z = (g_z)^{-1} (u_{az} - a_6 v_z), \quad g_z = (a_3 p_z + a_4) / a_5, \quad a_6 = (a_2 + a_1 a_3) / a_5, \quad e_z = (u_{az} - a_6 v_z) / g_z,$$

$$g_z = (a_3 p_z + a_4) / a_5, \quad a_6 = (a_2 + a_1 a_3) / a_5, \quad a_2 = \alpha_z - 0.5 r_z, \quad a_3 = D_z d_z + 0.5 a_2,$$

$$a_4 = D_z k_z + a_2 b_z, \quad a_5 = \alpha_z \exp(r_v L_z), \quad m_z = p_z e_z + a_1 v_z.$$

Now the initial-boundary value 2-D problem is in following form

$$\left\{ \begin{aligned} \frac{\partial v_z}{\partial t} &= \frac{\partial}{\partial x} \left(D_x \frac{\partial v_z}{\partial x} \right) + \frac{\partial}{\partial y} \left(D_y \frac{\partial v_z}{\partial y} \right) + a_{0z}^2 (u_{az} - a_6 v_z) - D_z r_v^2 v_z, \\ \frac{\partial v_z(0, y, t)}{\partial x} &= \frac{\partial v_z(x, 0, t)}{\partial y} = 0, \\ D_x \frac{\partial v_z(L_x, y, t)}{\partial x} &+ \alpha_x (v_z(L_x, y, t) - C_v u_{ax}) = 0, \\ D_y \frac{\partial v_z(x, L_y, t)}{\partial y} &+ \alpha_y (v_z(x, L_y, t) - C_v u_{ay}) = 0, \\ v_z(x, y, 0) &= C_v u 0(x, y), \quad x \in (0, L_x), \quad y \in (0, L_y), \quad t \in (0, t_f), \end{aligned} \right. \tag{7}$$

where $a_{0z}^2 = (2 D_z k_z) / L_z g_z, \quad C_v = (L_z)^{-1} \int_0^{L_z} \exp(r_v, z) dz = (\exp(r_v, L_z) - 1) / r_v, L_z.$

3.2. The CAM in y-direction

Using the averaged method with respect to y we use

$$v_y(x, t) = (L_y)^{-1} \int_0^{L_y} v_z(x, y, t) dy.$$

For the following hyperbolic type spline approximation

$$v_z(x, y, t) = v_y(x, t) + m_y(x, t)f_{y1} + e_y(x, t)f_{y2},$$

we have

$$f_{y1} = \frac{0.5L_y \sinh(a_y(y - 0.5L_y))}{\sinh(0.5a_yL_y)}, f_{y2} = \frac{\cosh(a_y(y - 0.5L_y)) - A_{0y}}{8 \sinh^2(0.25a_yL_y)},$$

where $A_{0y} = \frac{\sinh(0.5a_yL_y)}{0.5a_yL_y},$

and as the parameter we choose

$$a_y = a_{0z} \sqrt{a_6 / D_y} + r_v \sqrt{D_z / D_y}$$

$$a_y = a_{0z} \sqrt{a_6 / D_y} + r_v \sqrt{D_z / D_y}.$$

Similarly, we determine the unknown functions m_z, e_z from the boundary conditions by $z = 0, z = L_z$:

$$e_y = (u_{ay}C_v - v_y) / g_y, g_y = b_y + 0.5p_yL_y + 2k_yD_y / \alpha_y, m_y = p_y e_y, p_y = k_y / d_y,$$

$$d_y = 0.5a_yL_y \coth(0.5a_yL_y), k_y = 0.25a_y \coth(0.25a_yL_y), b_y = \frac{\cosh(0.5a_yL_y) - A_{0y}}{8 \sinh^2(0.25a_yL_y)}.$$

The initial-boundary value 1-D problem is in following form

$$\begin{cases} \frac{\partial v_y}{\partial t} = \frac{\partial}{\partial x} \left(D_x \frac{\partial v_y}{\partial x} \right) + a_{0y}^2 (u_{ay} - v_y) + a_{0z}^2 (u_{az} - a_6 v_y) - D_z r_v^2 v_y = 0, \\ \frac{\partial v_y(0,t)}{\partial x} = 0, D_x \frac{\partial v_y(L_x,t)}{\partial x} + \alpha_x (v_y(L_x,t) - C_v u_{ax}) = 0, \\ v_y(x,0) = C_v u_0(x), x \in (0, L_x), t \in (0, t_f), \end{cases} \tag{8}$$

where $a_{0y}^2 = (2D_y k_y) / L_y g_y, c_{0y}(x) = (L_y)^{-1} \int_0^{L_y} c_{0z}(x, y) dy.$

3.3. The CAM in x-direction

It is possible to proceed averaging also in x-direction

$$v_x(t) = (L_x)^{-1} \int_0^{L_x} v_y(x,t) dx.$$

For the following hyperbolic type spline approximation

$$v_y(x,t) = v_x(t) + m_x(t) f_{x1} + e_x(t) f_{x2},$$

we have

$$f_{x1} = \frac{0.5L_x \sinh(a_x(x - 0.5L_x))}{\sinh(0.5a_xL_x)}, f_{x2} = \frac{\cosh(a_x(x - 0.5L_x)) - A_{0x}}{8 \sinh^2(0.25a_xL_x)},$$

where $A_{0x} = \frac{\sinh(0.5a_xL_x)}{0.5a_xL_x},$

and as the parameter we choose

$$a_x = \sqrt{(a_{0z}^2 a_6 + a_{0y}^2 + D_z r_v^2) / D_x}.$$

Similarly, we determine the unknown functions m_x, e_x from the boundary conditions by $x = 0, x = L_x$ and

$$e_y = (u_{ax}C_v - v_x) / g_x, g_x = b_x + 0.5p_xL_x + 2k_xD_x / \alpha_x, m_x = p_x e_x, p_x = k_x / d_x,$$

$$d_x = 0.5a_x L_x \coth(0.5a_x L_x), k_x = 0.25a_x \coth(0.25a_x L_x), b_x = \frac{\cosh(0.5a_x L_x) - A_{0x}}{8 \sinh^2(0.25a_x L_x)}.$$

From problem (8) the initial problem of linear ODEs follows

$$\begin{cases} \frac{\partial c_x(t)}{\partial t} = a_{0y}^2(u_{ay} - v_x(t)) + a_{0z}^2(u_{az} - a_6 v_x(t)) - D_z r_v^2 v_x(t) + a_{0x}^2(u_{ax} - v_x(t)) = 0, \\ v_x(0) = C_v u_0, t \in (0, t_f), \end{cases}$$

where $a_{0x}^2 = (2D_x k_x) / L_x g_x$.

The solution of this problem can be obtained with the classical methods.

For $u_0 = 0$ we have,

$$c_x(t) = (A_0 / B_0)(1 - \exp(-B_0 t)),$$

where $A_0 = a_{0y}^2 u_{ay} + a_{0z}^2 u_{az} + a_{0x}^2 u_{ax}$, $B_0 = a_{0y}^2 + a_6 a_{0z}^2 + a_{0x}^2 + D_z r_v^2$.

In the stationary case we have

$$c_x = A_0 / B_0.$$

For fixed t_f we have:

$$v_y(x, t_f) = v_x(t_f) + m_x(t_f) f_{x1} + e_x(t_f) f_{x2}, e_x(t_f) = (u_{ax} - v_x(t_f)) / g_x, m_x(t_f) = p_x e_x(t_f),$$

$$v_z(x, y, t_f) = v_y(x, t_f) + m_y(x, t_f) f_{y1} + e_y(x, t_f) f_{y2}, e_y(x, t_f) = (u_{ay} - v_y(x, t_f)) / g_y.$$

$$m_y(x, t_f) = p_y e_y(x, t_f), v(x, y, z, t_f) = v_z(x, y, t_f) + m_z(x, y, t_f) f_{z1} + e_z(x, y, t_f) f_{z2},$$

$$e_z(x, y, t_f) = (u_{az} - a_6 v_z(x, y, t_f)) / g_z, m_z(x, y, t_f) = p_z e_z(x, y, t_f) + a_1 v_z(x, y, t_f),$$

$$u(x, y, z, t_f) = \exp(-r_v z) v(x, y, z, t_f).$$

4. Numerical approximations with ADI method for solving the initial boundary-value 3-D problem

We use uniform grid in the space $((K+1) \times (\tilde{N}+1) \times (M+1))$:

$$\begin{aligned} \{(z_k, y_i, x_j), z_k = (k-1) \cdot h_z, y_i = (i-1) \cdot h_y, x_j = (j-1) \cdot h_x, i = \overline{1, \tilde{N}+1}, \\ j = \overline{1, M+1}, k = \overline{1, K+1}, K \cdot h_z = L_z, \tilde{N} \cdot h_y = L_y, M \cdot h_x = L_x\}. \end{aligned}$$

For the time t we use the moments $t_n = n\tau$, $n = 0, 1, \dots$, subscripts (k, i, j, n) refer to z, y, x, t indices with the mesh spacing and for approximation of the function $u(z, y, x, t)$ we have the grid function with values

$$U_{k,i,j}^n \approx u(z_k, y_i, x_j, t_n).$$

For solving the 3-D problem (1) we use the discrete approximation

$$(U_{k,i,j}^{n+1} - U_{k,i,j}^n) / \tau = (\Lambda_z + \Lambda_y + \Lambda_x) U_{k,i,j}^{n+1}, n \geq 0, k = \overline{1, K+1}, i = \overline{1, \tilde{N}+1}, j = \overline{1, M+1}$$

And ADI method by J. Douglas and H. H. Rachford [19]:

$$\begin{cases} (U_{k,i,j}^{n+1/3} - U_{k,i,j}^n) / \tau = \Lambda_z U_{k,i,j}^{n+1/3} + \Lambda_y U_{k,i,j}^n + \Lambda_x U_{k,i,j}^n + f_{i,j}^n, k = \overline{2, K}, i = \overline{2, \tilde{N}}, j = \overline{2, M}, \\ (U_{k,i,j}^{n+2/3} - U_{k,i,j}^{n+1/3}) / \tau = \Lambda_y (U_{k,i,j}^{n+2/3} - U_{k,i,j}^{n+1/3}), k = \overline{1, K+1}, i = \overline{2, \tilde{N}}, j = \overline{1, M+1}, \\ (U_{k,i,j}^{n+1} - U_{k,i,j}^{n+2/3}) / \tau = \Lambda_x (U_{k,i,j}^{n+1} - U_{k,i,j}^{n+2/3}), k = \overline{1, K+1}, i = \overline{1, \tilde{N}+1}, j = \overline{2, M}. \end{cases} \quad (9)$$

After eliminating the fractional time moments $t_{n+1/3}, t_{n+2/3}$, we obtain the previous discrete problem with the approximation error $O(\tau^2)$. Here $\Lambda_x, \Lambda_y, \Lambda_z$ are the discrete difference operators, approximated expressions

$$\frac{\partial}{\partial x} \left(D_x \frac{\partial u(x, y, z, t)}{\partial x} \right), \frac{\partial}{\partial y} \left(D_y \frac{\partial u(x, y, z, t)}{\partial y} \right), \frac{\partial}{\partial z} \left(D_z \frac{\partial u(x, y, z, t)}{\partial z} \right) + r_z \frac{\partial u(x, y, z, t)}{\partial z}$$

with respect to x, y, z and boundary conditions with central differences.

For solving $U^{n+1/3}, U^{n+2/3}$ and U^{n+1} we use Thomas algorithm in z, y and x directions, respectively.

5. Results and discussion

This chapter checks the accuracy of the calculations with the CAM, using the ADI method.

Two solutions to the 3-D problem (1) are compared, one of which was obtained with the CAM using the above mentioned transformation, while the second solution was obtained with the ADI method (see Chapter 4).

The numerical results are obtained for the number of the grid points for (x, y, z) , ($\tilde{N} = M = 21, K = 19$) and for the following dimensionless parameters of BVP (1):

$$L_z = 1, L_y = 2, L_x = 3, \alpha_z = 200, \alpha_x = \alpha_y = 20, c_{az} = 1, c_{ax} = 3, c_{ay} = 2,$$

$$D_y = 0.3 \cdot 10^{-2}, D_x = 0.3 \cdot 10^{-2}, D_z = 10^{-2}, r_z = 0.1.$$

When using the ADI method, the stationary averaged solution $u_z(x, y, z, t_f)$ was chosen as the initial condition. During proceeding of the ADI method, a stationary solution was obtained with $\tau = 1, t_f = 300$ and with the maximal error (difference between two adjacent iterations) 10^{-8} .

For comparison of both of the presented methods the maximal values of the obtained solutions are the following: $u(x, y, 0, t_f) = 2.9956$ for averaged method (Fig. 3) and $u(x, y, 0, t_f) = 2.9964$ for ADI method (Fig. 4). Depending on the number of the grid points, (\tilde{N}, M, K) we have following maximal values for the ADI method: 2.9964 ($\tilde{N} = M = 21, K = 19$), 2.9963 ($\tilde{N} = M = 31, K = 29$).

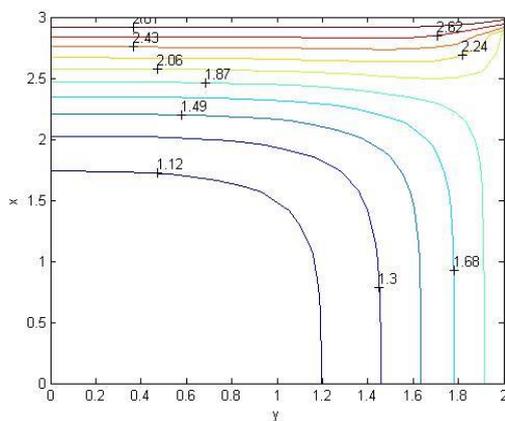


Fig. 3. Levels of averaged solution $u(x, y, 0, t_f)$

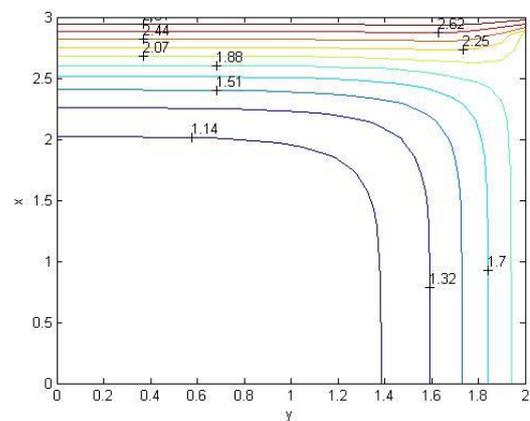


Fig. 4. Levels of numerical solution $u(x, y, 0, t_f)$

It appears that the CAM, in terms of precision, is perfectly similar to the ADI method, which requires both an adequate theoretical knowledge base on the method itself and adequate knowledge and skills for numerical execution of its algorithm.

It should also be noted that the usage of the CAM shows the possibility of acquiring new knowledge: the newly designed algorithms take the form of resolving previously studied tasks/problems in another way, which is different from traditional algorithms, and are often even simpler and easier to execute.

6. Conclusions

1. The present paper deals with the special transformation created that transforms the 3-D heat transfer boundary-value problem (BVP) with convection in z -direction to a corresponding problem (BVP) without convection.
2. Applying the method, the 1-D initial-boundary value problem (IBVP) with convection is solved using the special transformation – thus obtaining the IBVP without a convective term, which allowed to determine the coefficients of the spline function in such a way that the calculation error is minimal.
3. The corresponding 3-D BVP with the special transformation was also modified into a problem without a convective term, and further, based on the CAM, the problem obtained was reduced to 2-D and 1-D problems, using the integral hyperbolic splines with the parameters (the spline function coefficients) found.
4. The stationary numerical solution is compared with the spline solution. The maximal absolute value of the difference between the corresponding numerical and averaged solutions was 0.1-0.2 percent.
5. The methodology with the transformation described above allows us to solve also the tasks of a wider class – with convection also in y -direction and x -direction, thereby reducing the studying BVP to the problem without convective terms, which, in turn, allows us to determine the spline function coefficients for minimizing the calculation errors.

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